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On Linear Difference and Differential Equations.[*]

By Clyde E. Love.

I. *Introduction.*

1. In two papers[†] appearing in 1899 and in 1900 respectively, Dini developed certain general theorems on linear differential equations, and applied them to the integration of such equations for large values of the independent variable. Some years later Ford[‡] carried out the parallel investigation for linear difference equations. In the present paper an effort will be made to adapt the methods of Dini and Ford to the study of somewhat more general classes of equations than were considered in the above-mentioned investigations. It will therefore be well, for purposes of comparison, to state briefly the nature of these earlier results.

2. Dini's work may be summarized as follows:

In the differential equation

$$y^{(n)} + [a_1 + \alpha_1(x)]y^{(n-1)} + \dots + [a_n + \alpha_n(x)]y = 0, \tag{1}$$

suppose that the roots $\mu_1, \mu_2, \dots, \mu_n$ of the characteristic equation

$$\mu^n + a_1\mu^{n-1} + \dots + a_n = 0$$

are distinct. Let $\tau(x)$[§] be a suitably chosen positive function of x for which the integral

$$T(x) = \int_x^\infty \tau(t)\,dt$$

exists, and suppose that the functions $\alpha_1(x), \alpha_2(x), \dots, \alpha_n(x)$, in addition to satisfying certain conditions as to continuity, etc., have the property that

$$|\alpha_r^{(s)}(x)| \leqq \tau(x), \qquad r = 1, 2, \dots, n; \; s = 0, 1, \dots, n.$$

Then, if the roots $\mu_1, \mu_2, \dots, \mu_n$ all have the *same real part* $\bar{\mu}$, the general integral of (1) may be written, for large real and positive values of x, in the form

$$y = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x} + \dots + c_n e^{\mu_n x} + e^{\bar{\mu} x}\varepsilon(x),$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants, while $\varepsilon(x)$ vanishes at infinity to at least as high an order as does $T(x)$.

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[*] Read before the American Mathematical Society, April 2, 1915.

[†] *Annali di Matematica*, Ser. 3, Vol. II (1899), pp. 297–324; *ibid.*, Vol. III (1900), pp. 125–183.

[‡] *Annali di Matematica*, Ser. 3, Vol. XIII (1907), pp. 263–328.

[§] In Dini's notation, $\frac{T(x)}{x}$.

In case the roots are not all distinct, but all have the same real part, the behavior of the general solution is determined under suitably changed hypotheses regarding the functions $\alpha_1(x)$, $\alpha_2(x)$, ..., $\alpha_n(x)$.

Unfortunately the assumption that all the roots shall have the same real part diminishes greatly the range of applicability of these results. This assumption is essential to the method used by Dini, the method serving in the general case merely to determine the behavior of certain particular solutions.

3. Ford's results are in substance as follows:

In the difference equation

$$\mathbf{a}_0(x)y(x+n)+\mathbf{a}_1(x)y(x+n-1)+\ldots\ldots+\mathbf{a}_n(x)y(x)=0, \qquad (2)$$

let us write the coefficients in the form

$$\mathbf{a}_r(x)=a_r+\alpha_r(x), \qquad r=0,1,\ldots\ldots,n,$$

where $a_0\neq 0$, $a_n\neq 0$, and suppose that the roots of the characteristic equation

$$a_0\mu^n+a_1\mu^{n-1}+\ldots\ldots+a_n=0$$

are distinct. Let $\tau(x)$* be a suitably chosen positive function of x such that the series

$$T(x)=\sum_{t=x+1}^{\infty}\tau(t)$$

converges, and suppose that each of the functions $\alpha_0(x)$, $\alpha_1(x)$, ..., $\alpha_n(x)$ has the property, for sufficiently large values of x, that

$$|\alpha_r(x)|<\tau(x), \qquad r=0,1,\ldots\ldots,n.$$

Then, if the roots μ_1, μ_2, ..., μ_n all have the *same modulus* $\bar{\mu}$, the general solution of (2) takes for sufficiently large positive integral values of x the form

$$y(x)=c_1\mu_1^x+c_2\mu_2^x+\ldots\ldots+c_n\mu_n^x+\bar{\mu}^x\,\varepsilon(x),$$

where c_1, c_2, ..., c_n are arbitrary constants, while $\varepsilon(x)$ is infinitesimal in $1/x$ of an order at least as high as that of $T(x)$.

For the case of multiple roots, all having the same modulus, the analogous result is also obtained.

When the roots do not all have the same modulus, the method gives only the behavior of certain particular solutions.

4. The first part of the present paper is devoted to linear difference equations, the problem treated being closely related to the work of Ford just described. The results obtained constitute an advance over those of Ford in three principal respects: the restriction that the roots of the characteristic equation shall have the same modulus is dispensed with; the behavior of the

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* In Ford's notation, $\frac{\tau(x)}{x}$.

n fundamental integrals is determined separately; and the results are applied to the integration of the non-homogeneous equation.

The second part of the paper, dealing with linear differential equations, is similarly related to the work of Dini. The results of this part likewise possess the three characteristics just mentioned. The behavior of the derivatives of the integrals is also discussed.

The results of both Ford and Dini for the case of distinct roots fall out as special cases of certain theorems which Horn* has obtained by a somewhat different method. We shall therefore be chiefly concerned in the following pages with the case of multiple roots.

PART I. LINEAR DIFFERENCE EQUATIONS.

II *The Fundamental Theorems.*

5. As a preliminary step we shall develop two general theorems on linear difference equations.†

In the difference equation

$$a_0(x)y(x+n)+a_1(x)y(x+n-1)+\ldots.+a_n(x)y(x)=0, \tag{3}$$

suppose that the coefficients are defined for all integral values of x in the interval $x \geqq x_0$, and that $a_n(x)$ never vanishes in the same interval.

Let us choose n *auxiliary functions* $z_1(x), z_2(x), \ldots., z_n(x)$, each defined in the interval $x \geqq x_0-n$, and having the property that the determinant

$$Q(x)=\begin{vmatrix} z_1(x) & z_1(x-1) & \ldots. & z_1(x-n+1) \\ z_2(x) & z_2(x-1) & \ldots. & z_2(x-n+1) \\ \ldots & \ldots & \ldots & \ldots \\ z_n(x) & z_n(x-1) & \ldots. & z_n(x-n+1) \end{vmatrix}$$

never vanishes in that interval.

Let $A_r(x)$ be the minor of $Q(x)$ with respect to the element $z_r(x)$. Put

$$Z_r(x)=z_r(x)a_n(x)+z_r(x-1)a_{n-1}(x-1)+\ldots.+z_r(x-n)a_0(x-n), \qquad r=1, \ldots., n,$$

and denote by $q(x, t)$ the determinant formed from $Q(x)$ by replacing $z_r(x)$ by $Z_r(t)$, $r=1, \ldots., n$. Also make the following definitions:

$$f_r(x)=\frac{(-1)^n C_r A_r(x)}{a_n(x)\,Q(x)}, \qquad r=1, \ldots., n,$$

where $C_1, C_2, \ldots., C_n$ are arbitrary constants,

$$K(x, t)=\frac{(-1)^n q(x, t)}{a_n(x)\,Q(x)},$$

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[*] *Journal für Mathematik,* Vol. CXXXVIII (1910), pp. 159–191.

[†] These theorems differ only slightly from certain theorems given by Ford (*loc. cit.*).

$$u_{r,0}(x) = v_{r,0}(x) = f_r(x), \tag{4}$$

$$u_{r,m}(x) = \sum_{t=x+1}^{\infty} K(x,t)\, u_{r,m-1}(t), \qquad m=1, 2, \ldots\ldots, \tag{5}$$

$$v_{r,m}(x) = \sum_{t=x_0}^{x} K(x,t)\, v_{r,m-1}(t), \qquad m=1, 2, \ldots\ldots. \tag{6}$$

Then we have

THEOREM A. *If for all integral values of* $x \geqq x_0$

(a) *the series* $\sum_{m=0}^{\infty} |u_{i,m}(x)|$, *where i is one of the numbers* $1, \ldots\ldots, n$, *converges to a limit* $U_i(x)$ *such that*

(b) *the series* $\sum_{t=x+1}^{\infty} |K(x,t)|\, U_i(t)$ *converges also;*

(c) *the series*

$$y_i(x) = \sum_{m=0}^{\infty} u_{i,m}(x) \tag{7}$$

defines a function such that each of the series $\sum_{t=x+1}^{\infty} y_i(t)\, Z_r(t)$, $r=1, \ldots\ldots, n$, *converges,*

then $y_i(x)$ *as given by* (7) *is a solution of* (3) *for all values of* $x \geqq x_0+n-1$.

THEOREM B. *If the series*

$$y_i(x) = \sum_{m=0}^{\infty} v_{i,m}(x),$$

where i is one of the numbers $1, \ldots\ldots, n$, *is convergent, then* $y_i(x)$ *is a solution of* (3) *for* $x \geqq x_0+n-1$.

6. To prove Theorem A, we note first that (2) may be written in the form*

$$\alpha_0(x)\, \Delta^n y(x) + \alpha_1(x)\, \Delta^{n-1} y(x+1) + \ldots\ldots + \alpha_n(x)\, y(x+n) = 0,$$

where

$$\alpha_r(x) = \frac{(-1)^{n-r}}{(n-r)\,!} \sum_{i=0}^{r} \frac{(n-i)\,!}{(r-i)\,!} a_{n-i}(x), \qquad r=0, 1, \ldots\ldots, n.$$

Now we have at once

$$y_i(x) = f_i(x) + \sum_{m=0}^{\infty} \sum_{t=x+1}^{\infty} K(x,t)\, u_{i,m}(t).$$

In view of hypothesis (b), we may invert the order of summation, and write

$$y_i(x) = f_i(x) + \sum_{t=x+1}^{\infty} K(x,t) \sum_{m=0}^{\infty} u_{i,m}(t)$$

$$= f_i(x) + \sum_{t=x+1}^{\infty} K(x,t)\, y_i(t). \tag{8}$$

Next, let us use the symbols z_r, α_s, $p_{r,m}$, $q_{r,m}$ to denote the functions $z_r(x-n+1)$, $\alpha_s(x-n+1)$, $p_{r,m}(x)$, $\Delta p_{r,m}(x)$ respectively, and place

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* Cf. Ford, *loc. cit.*

$$p_{r,0}=z_r\alpha_0,\ p_{r,1}=z_r\alpha_1-q_{r,0},\ \ldots\ldots,\ p_{r,n-1}=z_r\alpha_{n-1}-q_{r,n-2},\qquad r=1,\ldots\ldots,n;$$

$$\Phi_r(x)=\sum_{t=x+1}^{\infty}Z_r(t)\,y_i(t),\qquad r=1,\ldots\ldots,i-1,\,i+1,\ldots\ldots,n;$$

$$\Phi_i(x)=\sum_{t=x+1}^{\infty}Z_i(t)\,y_i(t)+C_i;$$

$$P(x)=\begin{vmatrix} p_{1,n-1} & p_{1,n-2} & \cdots\cdots & p_{1,0}\\ p_{2,n-1} & p_{2,n-2} & \cdots\cdots & p_{2,0}\\ \cdots & \cdots & \cdots & \cdots\\ p_{n,n-1} & p_{n,n-2} & \cdots\cdots & p_{n,0}\end{vmatrix},\quad P_i(x)=\begin{vmatrix} \Phi_1(x) & p_{1,n-2} & \cdots\cdots & p_{1,0}\\ \Phi_2(x) & p_{2,n-2} & \cdots\cdots & p_{2,0}\\ \cdots & \cdots & \cdots & \cdots\\ \Phi_n(x) & p_{n,n-2} & \cdots\cdots & p_{n,0}\end{vmatrix}$$

Upon substituting the values of $f_i(x)$ and $K(x,t)$ in (8), we find after certain transformations that

$$y_i(x)=\frac{P_i(x)}{P(x)}.$$

Consider now the system of n functions $\eta_0,\ \eta_1,\ \ldots\ldots,\ \eta_{n-1}$, each defined for all values of $x\geqq x_0$ by the system of equations

$$p_{r,n-1}\eta_0+p_{r,n-2}\eta_1+\ldots\ldots+p_{r,0}\eta_{n-1}=\Phi_r(x),\qquad r=1,\ldots\ldots,n. \tag{9}$$

We have at once

$$\eta_0(x)=y_i(x).$$

Further, let us take the first difference of each member of (9) and write ζ_s for $\Delta\eta_s(x)$, thus:

$$\begin{aligned}p_{r,n-1}\zeta_0+p_{r,n-2}\zeta_1+\ldots\ldots+p_{r,0}\zeta_{n-1}+q_{r,n-1}\eta_0(x+1)+q_{r,n-2}\eta_1(x+1)&\\ +\ldots\ldots+q_{r,0}\eta_{n-1}(x+1)=-y_i(x+1)Z_r(x+1).&\end{aligned} \tag{10}$$

But $y_i(x+1)=\eta_0(x+1)$, and $Z_r(x+1)=z_r\alpha_n-q_{r,n-1}$. Equation (10) thus becomes

$$\begin{aligned}&z_r\alpha_0\zeta_{n-1}+(z_r\alpha_1-q_{r,0})\,\zeta_{n-2}+\ldots\ldots+(z_r\alpha_{n-1}-q_{r,n-2})\zeta_0\\ &\quad+(z_r\alpha_n-q_{r,n-1})\eta_0(x+1)+q_{r,0}\,\eta_{n-1}(x+1)+q_{r,1}\,\eta_{n-2}(x+1)\\ &\quad+\ldots\ldots+q_{r,n-1}\,\eta_0(x+1)=0,\end{aligned}$$

or

$$\theta z_r+\theta_1 q_{r,0}+\ldots\ldots+\theta_{n-1}q_{r,n-2}=0,\qquad r=1,\ldots\ldots,n, \tag{11}$$

where

$$\theta=\alpha_0\zeta_{n-1}+\alpha_1\zeta_{n-2}+\ldots\ldots+\alpha_{n-1}\zeta_0+\alpha_n\eta_0(x+1),$$

and

$$\theta_s=\eta_{n-s}(x+1)-\zeta_{n-s-1},\qquad s=1,\ldots\ldots,n-1. \tag{12}$$

The system (11) is a set of n homogeneous linear equations in $\theta,\theta_1,\ldots\ldots,\theta_{n-1}$. Since $q_{r,s-1}=z_r\alpha_s-p_{r,s}$, it appears that the discriminant of the system does not vanish. Hence

$$\theta\equiv\theta_1\equiv\ldots\ldots\equiv\theta_{n-1}\equiv 0. \tag{13}$$

We shall now show that

$$\eta_s(x)=\Delta^s y_i(x-s). \tag{14}$$

Assuming that

$$\eta_{s-1}(x)=\Delta^{s-1}y_i(x-s+1), \tag{15}$$

we find

$$\Delta\eta_{s-1}(x) = \Delta^s y_i(x-s+1).$$

But since

$$\theta_{n-s} = \eta_s(x+1) - \Delta\eta_{s-1}(x) = 0,$$

we have

$$\eta_s(x+1) = \Delta^s y_i(x-s+1),$$

or

$$\eta_s(x) = \Delta^s y_i(x-s). \tag{16}$$

Since the assumption (15) is true for $s=1$, formula (14) is established; whence, taking the first difference of each member of (16), we obtain

$$\Delta\eta_s(x) = \Delta^{s+1} y(x-s). \tag{17}$$

Now upon substituting in θ the values of $\Delta\eta_{n-1}(x), \ldots, \Delta\eta_0(x)$ as obtained from (12), (13) and (17), we find

$$\alpha_0 \Delta^n y_i(x-n+1) + \alpha_1 \Delta^{n-1} y_i(x-n+2) + \ldots + \alpha_n y_i(x+1) = 0,$$

or, for values of $x \geqq x_0+n-1$,

$$\alpha_0(x)\,\Delta^n y_i(x) + \alpha_1(x)\,\Delta^{n-1} y_i(x+1) + \ldots + \alpha_n(x)\,y_i(x+n) \equiv 0,$$

which was to be proved.

The proof of Theorem B may be carried out similarly.

III. *The Homogeneous Equation: Distinct Roots.**

7. In the difference equation

$$Y(x) = \mathfrak{a}_0(x)y(x+n) + \mathfrak{a}_1(x)y(x+n-1) + \ldots + \mathfrak{a}_n(x)y(x) = 0, \tag{2}$$

wherein

$$\mathfrak{a}_r(x) = a_r + \alpha_r(x), \qquad r=0, 1, \ldots, n,$$

let us assume that $a_0 \neq 0$, $a_n \neq 0$, and that the roots $\mu_1, \mu_2, \ldots, \mu_n$ of the characteristic equation

$$\phi(\mu) = a_0\mu^n + a_1\mu^{n-1} + \ldots + a_n = 0 \tag{18}$$

are distinct.

Suppose further that a positive function $\tau(x)$ exists such that

$$|\alpha_r(x)| \leqq \tau(x), \qquad r=0, 1, \ldots, n,$$

where $\tau(x)$ has the following properties:

(a) The series $T(x) = \sum_{t=x+1}^{\infty} \tau(t)$ is convergent;

(b) the ratio $\frac{x\tau(x)}{T(x)}$ remains finite for large values of x;

(c) the function $e^{ax}\tau(x)$ increases monotonically with x for all values of $a>0$.

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* As noted above, the results for this case are hardly new (cf. Horn, *loc. cit.*). However, it is believed that the treatment from the standpoint of Ford's theory may be of some interest. Further, having in mind the work of this section, we are enabled to omit many details in the discussion of the more complicated case following.

These properties are evidently possessed by each of the functions

$$\frac{1}{x^{\nu}},\ \frac{1}{x(\log x)^{\nu}},\ \frac{1}{x\log x\,(\log\log x)^{\nu}},\ \dots\dots, \qquad \nu>1.$$

For simplicity we shall assume that $\tau(x)$ may be taken as one of this set of functions.

8. Let

$$\delta=\begin{vmatrix} 1 & \mu_1 & \dots\dots & \mu_1^{n-1} \\ 1 & \mu_2 & \dots\dots & \mu_2^{n-1} \\ \dots & \dots & \dots\dots & \dots \\ 1 & \mu_n & \dots\dots & \mu_n^{n-1} \end{vmatrix}=\prod_{r,\,s}(\mu_r-\mu_s), \qquad r>s,$$

and denote by δ_r the cofactor of the element μ_r^{n-1}. Also put

$$\mu=\mu_1\mu_2\dots\dots\mu_n,$$
$$\sigma_{1,r}=\mu_r^n\alpha_0(x-n)+\mu_r^{n-1}\alpha_1(x-n+1)+\dots\dots+\alpha_n(x).$$

With this notation we find

$$Q(x)=\delta\mu^{-x},$$
$$f_r(x)=(-1)^n C_r\delta_r\mu_r^x/\delta\mathbf{a}_n(x), \qquad r=1,\dots\dots,n, \tag{19}$$
$$Z_r(x)=z_r(x)\sigma_{1,r}(x),$$
$$K(x,t)=\sum_{r=1}^{n}\delta_r\mu_r^{x-t}\sigma_{1,r}(t)/\delta\mathbf{a}_n(x). \tag{20}$$

We note that a constant M_1 can be found so large that

$$|\sigma_{1,r}(x)|\leqq M_1\tau(x), \qquad r=1,\dots\dots,n. \tag{21}$$

Let our notation be such that

$$|\mu_1|\leqq|\mu_2|\leqq\dots\dots\leqq|\mu_n|, \tag{22}$$

and consider first the case in which these quantities $|\mu_r|$ are all distinct.

We have

$$|K(x,t)\,f_1(t)|\leqq\sum_{r=1}^{n}|\delta_1\delta_r C_1\mu_r^{x-t}\mu_1^t\sigma_{1,r}(t)/\delta^2\mathbf{a}_n(x)\,\mathbf{a}_n(t)|$$
$$\leqq|f_1(x)|\sum_{r=1}^{n}\left|\delta_r\left(\frac{\mu_1}{\mu_r}\right)^{t-x}\sigma_{1,r}(t)\ \Big/\delta\mathbf{a}_n(t)\right|.$$

Now since $|\mu_1|\leqq|\mu_r|$, $r=1,\dots\dots,n$, it is evident that the quantity

$$\left|\delta_r\left(\frac{\mu_1}{\mu_r}\right)^{t-x}\Big/\delta\mathbf{a}_n(t)\right|$$

remains less than some fixed constant for all values of x and t in question. Hence, by (21), a constant M_2 can be found such that

$$|K(x,t)\,f_1(t)|<M_2|f_1(x)|\sum_{r=1}^{n}\frac{|\sigma_{1,r}(t)|}{M_1}$$
$$<nM_2|f_1(x)|\,|\tau(t)|. \tag{23}$$

9. We are now in position to show that the conditions of Theorem A are satisfied by the function

$$y_1(x) = \sum_{m=0}^{\infty} u_{1,m}(x), \tag{24}$$

where $u_{1,m}(x)$, $f_1(x)$ and $K(x,t)$ are given by (4) and (5), (19), and (20) respectively.

We shall first prove by mathematical induction that

$$|u_{1,m}(x)| \leqq [M_2 n T(x)]^m |f_1(x)|, \qquad m=0,1,2,\ldots\ldots \tag{25}$$

Assume that this formula is true for $u_{1,m-1}(x)$. Then

$$\begin{aligned}|u_{1,m}(x)| &\leqq \sum_{t=x+1}^{\infty} |K(x,t) f_1(t)| \, [M_2 n T(t)]^{m-1} \\ &\leqq [M_2 n T(x)]^{m-1} \sum_{t=x+1}^{\infty} |K(x,t) f_1(t)|,\end{aligned}$$

since $T(x)$ is a monotonically decreasing function. By (23), we have at once the relation (25). To complete the proof by induction, we need only point out that our assumption regarding $u_{m-1}(x)$ is justified, by (4), for $m=1$.

Let x now be restricted to values so large that

$$M_2 n T(x) \leqq \theta, \qquad 0<\theta<1,$$

where θ is a constant independent of x. Then, as a consequence of (25), condition (a) of Theorem A is satisfied, by Weierstrass's test; and further, we find

$$U_1(x) = \sum_{m=0}^{\infty} |u_{1,m}(x)| \leqq \frac{|f_1(x)|}{1-M_2 n T(x)} \leqq \frac{|f_1(x)|}{1-\theta}. \tag{26}$$

As regards (b), we have

$$\begin{aligned}\sum_{t=x+1}^{\infty} |K(x,t)| U_1(t) &\leqq \frac{1}{1-\theta} \sum_{t=x+1}^{\infty} |K(x,t) f_1(t)| \\ &\leqq \frac{M_2 n}{1-\theta} |f_1(x)| \, T(x) \leqq \frac{\theta}{1-\theta} |f_1(x)|,\end{aligned}$$

so that this condition is satisfied also.

From (19), (21) and (26) it follows at once that (c) holds.

Thus $y_1(x)$ as given by (24) is a solution of (2) for large values of x. As a consequence of (25) we have

$$\begin{aligned}|y_1(x) - f_1(x)| &\leqq \sum_{m=1}^{\infty} |u_{1,m}(x)| \\ &\leqq \frac{n M_2 T(x) \, |f_1(x)|}{1-M_2 n T(x)};\end{aligned}$$

whence we may write

$$y_1(x) = \mu_1^x [1+\varepsilon_1(x)],$$

where $\varepsilon_1(x)$ vanishes, when $x=\infty$, to at least as high an order as that of $T(x)$.

10. In the above argument, the convergence of the series defining $u_{1,m}(x)$ depends on the fact that $|\mu_1| \leqq |\mu_r|$, $r=1, \ldots, n$. When we try to find a solution $y_2(x)$ corresponding to the root μ_2, the process breaks down by reason of the fact that $|\mu_2| > |\mu_1|$. We may avoid the difficulty by the use of Theorem B, as will now be shown.

The difference-equation

$$Z(x) = \mathbf{a}_n(x) z(x) + \mathbf{a}_{n-1}(x-1) z(x-1) + \ldots + \mathbf{a}_0(x-n) z(x-n) = 0 \tag{27}$$

evidently satisfies all the conditions that have been imposed upon equation (2). The roots of its characteristic equation

$$a_n \mu^n + a_{n-1}\mu^{n-1} + \ldots + a_0 = 0$$

are $\frac{1}{\mu_n}, \frac{1}{\mu_{n-1}}, \ldots, \frac{1}{\mu_1}$. Therefore, by the result of the preceding paragraph, equation (27) has a solution corresponding to the root $\frac{1}{\mu_n}$ of the form

$$z_n(x) = \mu_n^{-x}[1+\eta_n(x)], \tag{28}$$

where $\eta_n(x)$ has the property assigned above to $\varepsilon_1(x)$.

Let us now choose a set of auxiliary functions in which $z_1, z_2, \ldots, z_{n-1}$ are as before, while z_n is given by (28). With this choice of z_n, we have

$$Z_n(x) \equiv 0.$$

The functions $f_r(x)$ and $K(x,t)$ are not materially changed, except that in $K(x,t)$ the summation now runs from $r=1$ to $r=n-1$.

Consider the function

$$y_n(x) = \sum_{m=0}^{\infty} v_{n,m}(x),$$

where $v_{n,m}(x)$ is given by (4) and (6). We find

$$K(x,t)\, f_n(t) = \sum_{r=1}^{n-1} C_n \mu_r^{x-t} \mu_n^t \rho_n(x)\, \sigma_{n,r}(t),$$

where $\rho_n(x)$ remains finite for large values of x, while a constant M_3 can be found such that

$$|\sigma_{n,r}(x)| \leqq M_3 \tau(x), \qquad r=1, \ldots, n.$$

Therefore a constant M_4 can be chosen so large that

$$|K(x,t)\, f_n(t)| \leqq |f_n(x)|\, M_4 \sum_{r=1}^{n-1} \left| \left(\frac{\mu_n}{\mu_r}\right)^{t-x} \right| \tau(t).$$

It can now be shown by mathematical induction that

$$|v_{n,m}(x)| \leqq [M_4(n-1)\, x\tau(x)]^m \, |f_n(x)|. \tag{29}$$

For, this is true for $v_{n,0}(x)$, by definition (cf. (4)). Assume it to hold for $v_{n,m-1}(x)$. Then

$$
\begin{aligned}
|v_{n,m}(x)| &\leqq \sum_{t=x_0}^{x} |K(x,t)\, v_{n,m-1}(t)| \\
&\leqq [M_4(n-1)]^{m-1} \sum_{t=x_0}^{x} |K(x,t)\, f_n(t)|\, [t\tau(t)]^{m-1} \\
&\leqq M_4^m (n-1)^{m-1} |f_n(x)| \sum_{t=x_0}^{x} \left\{ t^{m-1} [\tau(t)]^m \sum_{r=1}^{n-1} \left| \left(\frac{\mu_n}{\mu_r}\right)^{t-x} \right| \right\}. \qquad (30)
\end{aligned}
$$

Now, each of the functions $t^{m-1}[\tau(t)]^m \left| \left(\frac{\mu_n}{\mu_r}\right)^t \right|$, $r=1, \ldots, n-1$, increases monotonically with t. Thus we may replace each of these functions in the right member of (30) by its largest value, viz., its value when $x=t$, which gives (29) at once.

As a consequence of (29), we have

$$|y_n(x)| \leqq \sum_{m=0}^{\infty} |v_{n,m}(x)| \leqq \frac{|f_n(x)|}{1-M(n-1)\, x\tau(x)} < \frac{|f_n(x)|}{1-\theta},$$

provided x be taken so large that

$$M(n-1)\, x\tau(x) < \theta, \qquad 0<\theta<1.$$

Thus, for such values of x, Theorem B is satisfied, and $y_n(x)$ is a solution of (2). Further, by (29),

$$
\begin{aligned}
|y_n(x) - f_n(x)| &\leqq \sum_{m=1}^{\infty} |v_{n,m}(x)| \\
&\leqq \frac{M(n-1)\, x\tau(x)}{1-M(n-1)\, x\tau(x)},
\end{aligned}
$$

so that

$$y_n(x) = \mu_n^x [1+\varepsilon_n(x)], \qquad (31)$$

where $\varepsilon_n(x)$ is infinitesimal in $1/x$ to at least as high an order as that of $x\tau(x)$, and therefore, by hypothesis, to as high an order as that of $T(x)$.

11. To find a third particular solution of (2), we may proceed as follows. Equation (27) has, by (31), a solution

$$z_1(x) = \mu_1^{-x} [1+\eta_1(x)].$$

Introducing this in place of the original $z_1(x)$, thus making $Z_1(x) \equiv 0$, we find that, in $K(x,t)$, the summation runs only from $r=2$ to $r=n-1$. Proceeding exactly as in § 9, we find an integral $y_2(x)$ of (2) having the form

$$y_2(x) = \mu_2^x [1+\varepsilon_2(x)],$$

where $\varepsilon_2(x)$ has the usual property for large x.

Now if we introduce in place of $z_{n-1}(x)$ the solution

$$z_{n-1}(x) = \mu_{n-1}^x [1+\eta_{n-1}(x)]$$

of (27), we may obtain as in § 10 a solution of (2) of the form

$$y_{n-1}(x) = \mu_{n-1}^x [1+\varepsilon_{n-1}(x)].$$

The remainder of the process is obvious. Using Theorem A and Theorem B alternately, we obtain n linearly independent integrals of (2) having, for sufficiently large values of x, the form

$$y_i = \mu_i^x [1 + \varepsilon_i(x)], \qquad i = 1, \ldots, n. \tag{32}$$

12. In case the quantities $|\mu_r|$, $r = 1, \ldots, n$, are not all different, suppose they fall into l groups containing $n_1, n_2, \ldots, n_l$ roots respectively $(n_1 + n_2 + \ldots + n_l = n)$, in such a way that

$$|\mu_1| = |\mu_2| = \ldots = |\mu_{n_1}| < |\mu_{n_1+1}| = \ldots = |\mu_{n_1+n_2}| < \ldots < |\mu_{n-n_l+1}| = \ldots = |\mu_n|.$$

Then by using Theorem A as in § 9 we find n_1 independent integrals of (2) corresponding to that group of roots whose moduli are equal and least. With a properly revised system of auxiliary functions Theorem B gives n_l solutions corresponding to the group of roots with greatest modulus. Going on in this way, we obtain again (32).

In particular, if the roots all have the same modulus, which is the case treated by Ford, the complete result can be written down by the method of § 8.

IV. *The Homogeneous Equation: Multiple Roots.*

13. Suppose that the characteristic equation (18) has l distinct roots $\mu_1, \ldots, \mu_l$, occurring $n_1, \ldots, n_l$ times respectively: $n_1 + \ldots + n_l = n$. Let n' be the largest of the numbers $n_1, \ldots, n_l$. Further, let the coefficients in equation (2) be such that, when x is large,

$$|a_r(x)| < \tau(x)/x^{2n'-2}, \qquad r = 1, \ldots, n,$$

$\tau(x)$ having the same meaning as before.

Let us take n auxiliary functions of the form*

$$z_{r,q}(x) = x^{n_r - q} \mu_r^{-x}, \qquad r = 1, \ldots, l;\ q = 1, \ldots, n_r. \tag{33}$$

14. Place

$$\delta = \prod_{r,s} 0!\,1!\ldots(n_r - 1)!\,\mu_r^{\frac{1}{2} n_r (n_r - 1)} (\mu_r - \mu_s)^{n_r n_s}, \qquad r = 1, \ldots, l;\ r > s;$$

also

$$\mu = \mu_1^{n_1} \mu_2^{n_2} \ldots \mu_l^{n_l}.$$

Then we find

$$Q(x) = \delta \mu^{-x},$$

and

$$f_{r,q}(x) = x^{q-1} \mu_r^x \rho_{r,q}{}^{(x)}, \qquad r = 1, \ldots, l;\ q = 1, \ldots, n_r, \tag{34}$$

where $\rho_{r,q}(x)$ remains finite and different from 0 as $x = \infty$. Also

$$Z_{r,q}(x) = z_{r,q}(x)\, \sigma_{r,q}(x),$$

where a constant M_5 can be found such that

---

* For brevity we write $z_{n_1 + \ldots + n_{r-1} + q} = z_{r,q}$, with a similar meaning for $f_{r,q}$, etc.

$$|\sigma_{r,q}(x)| < M_5 \tau(x)/x^{2n'-2}.$$

Thus

$$K(x,t) = \sum_{r=1}^{l} \sum_{q=1}^{n_r} x^{q-1} \mu_r^x \nu_{r,q}^x t^{n_r-q} \mu_r^{-t} \sigma_{r,q}(t), \tag{35}$$

where $\nu_{q,r}(x)$ remains finite as $x = \infty$.

From (34) and (35) we have

$$\begin{aligned} |K(x,t) f_{1,k}(t)| &\leqq \sum_{r=1}^{l} \sum_{q=1}^{n_r} x^{q-1} t^{n_r-q+k-1} |\mu_r^{x-t} \mu_1^t \nu_{r,q}(x) \sigma_{r,q}(t) \rho_{1,k}(t)| \\ &\leqq M_6 |f_{1,k}(x)| \sum_{r=1}^{l} \sum_{q=1}^{n_r} x^{q-k} t^{n_r-q+k-2n'+1} \left| \left(\frac{\mu_1}{\mu_r}\right)^{t-x} \right| \tau(t), \quad k=1, \ldots, n_1, \end{aligned}$$

where M_6 is the maximum value, for all values of x and t in question, of the quantity $M_5 \left| \frac{\rho_{1,k}(t)}{\rho_{1,k}(x)} \nu_{r,q}(x) \right|$. This latter inequality may be written

$$|K(x,t) f_{1,k}(t)| \leqq M_6 |f_{1,k}(x)| \sum_{r=1}^{l} \sum_{q=1}^{n_r} x^{1-k} \left(\frac{x}{t}\right)^{q-1} t^{n_r+k-2n'} \left| \left(\frac{\mu_1}{\mu_r}\right)^{t-x} \right| \tau(t).$$

Since $|\mu_1| \leqq |\mu_r|$, $k \geqq 1$, $t > x$, $q \geqq 1$, $2n' \geqq n_r + k$, we have

$$|K(x,t) f_{1,k}(t)| \leqq M_6 n |f_{1,k}(t)| \tau(t). \tag{36}$$

15. It can now be shown that each of the functions

$$y_{1,k}(x) = \sum_{m=0}^{\infty} u_{1,k,m}(x), \qquad k=1, \ldots, n_1, \tag{37}$$

as given by (7) is a solution of (2) for large values of x.

We shall first prove by mathematical induction that

$$|u_{1,k,m}(x)| \leqq [M_6 n T(x)]^m |f_{1,k}(x)|, \tag{38}$$

where

$$T(x) = \sum_{t=x+1}^{\infty} \tau(t).$$

This is true of $u_{1,k,0}(x)$ by definition (cf. (4)). Assume it to hold for $u_{1,k,m-1}(x)$. Then

$$\begin{aligned} |u_{1,k,m}(x)| &\leqq \sum_{t=x+1}^{\infty} |K(x,t) u_{1,k,m-1}(t)| \\ &\leqq (M_6 n)^{m-1} \sum_{t=x+1}^{\infty} |K(x,t) f_{1,k}(t)| [T(t)]^{m-1} \\ &\leqq (M_6 n)^m |f_{1,k}(x)| \sum_{t=x+1}^{\infty} [T(t)]^{m-1} \tau(t), \end{aligned}$$

by (36). Since $T(x)$ decreases monotonically with x, we may write

$$|u_{1,k,m}(x)| \leqq (M_6 n)^m [T(x)]^{m-1} |f_{1,k}(x)| \sum_{t=x+1}^{\infty} \tau(t),$$

which is the desired relation.

Let us henceforth restrict ourselves to values of x so large that

$$M_6 n T(x) \leqq \theta, \qquad 0 < \theta < 1.$$

Then we have by (38)

$$U_{1,k}(x) = \sum_{m=0}^{\infty} |u_{1,k,m}(x)|$$
$$\leqq \frac{|f_{1,k}(x)|}{1-M_6 n T(x)} \leqq \frac{|f_{1,k}(x)|}{1-\theta}, \tag{39}$$

so that condition (a) of Theorem A is satisfied.

Further, we have by (39) and (36)

$$\sum_{t=x+1}^{\infty} |K(x,t)| U_{1,k}(t) \leqq \frac{1}{1-\theta} \sum_{t=x+1}^{\infty} |K(x,t) f_{1,k}(t)|$$
$$\leqq \frac{M_6 n}{1-\theta} |f_{1,k}(x)| T(x),$$

whence condition (b) is satisfied.

Finally, denoting by M_7 the maximum value of $M_5|\rho_{1,k}(x)|$, we find

$$|y_{1,k}(x) Z_{r,q}(x)| \leqq \frac{1}{1-\theta} |f_{1,k}(x)| x^{n_r-q+2-2n'} |\mu_r^{-x} \sigma_{r,q}(x)|$$
$$\leqq \frac{M_7}{1-\theta} \left| \left(\frac{\mu_1}{\mu_r}\right)^x \right| x^{n_r-q+2-2n'} \tau(x).$$

Since $|\mu_1| \leqq |\mu_r|$ and $2n' \geqq n_r - q + 2$, this shows that

$$\sum_{t=x+1}^{\infty} |y_{1,k}(t) Z_{r,q}(t)| \leqq \frac{M_7 T(x)}{1-\theta},$$

whence (c) is satisfied.

The proof that each of the functions $y_{1,k}(x)$ as given by (37) satisfies (2) for sufficiently large values of x is now complete. Further, as a consequence of (38) we have

$$|y_{1,k}(x) - f_{1,k}(x)| \leqq \sum_{m=1}^{\infty} |u_{1,k,m}(x)|$$
$$\leqq \frac{M_6 n T(x)}{1-M_6 n T(x)} |f_{1,k}(x)|,$$

whence we may write

$$y_{1,k}(x) = x^{k-1} \mu_1^x [1+\varepsilon_{1,k}(x)], \qquad k=1, \ldots, n_1, \tag{40}$$

where $\varepsilon_{1,k}(x)$ is infinitesimal in $1/x$ of at least as high an order as that of $T(x)$.

16. To obtain further particular solutions of (2), we shall proceed as in § 10. The equation $Z(x)=0$ (cf. (27)) has, by (40), a group of n_l solutions corresponding to the n-fold root $1/\mu_l$ of its characteristic equation

$$a_n \mu^n + \ldots + a_0 = 0,$$

which solutions may be expressed in the form

$$z_{l,q}(x) = x^{n_l-q} \mu_1^{-x} [1+\eta_{l,k}(x)], \qquad q=1, \ldots, n_l.$$

Let us introduce these functions into the auxiliary system (33) in place of the functions $z_{l,q}(x)$ as originally chosen, thus making

$$Z_{l,q} \equiv 0, \qquad q=1, \ldots, n_l.$$

Thus in $K(x,t)$ the summation with respect to r runs from $r=1$ to $r=l-1$ only.

We may now show by Theorem B, by much the same argument as that used in § 10, that (2) has a group of integrals

$$y_{l,k}(x) = x^{k-1}\mu_l^x[1+\varepsilon_{l,k}(x)], \qquad k=1, \ldots, n_l.$$

The remainder of the argument for the case where the quantities $|\mu_r|$ are distinct, and also for that where these quantities are not distinct, may be easily filled in by referring to §§ 12–13.

17. The results of this section may be summarized in

THEOREM I. *Let $\tau(x)$ be one of the system of functions*[*]

$$\frac{1}{x^\nu}, \ \frac{1}{x(\log x)^\nu}, \ \frac{1}{x\log x(\log\log x)^\nu}, \ \ldots, \qquad \nu>1,$$

and put

$$T(x) = \sum_{t=x+1}^{\infty} \tau(t).$$

In the difference equation

$$Y(x) = \mathfrak{a}_0(x)y(x+n)+\mathfrak{a}_1(x)y(x+n-1)+\ldots+\mathfrak{a}_n(x)y(x)=0,$$

where $\mathfrak{a}_r(x)=a_r+\alpha_r(x)$, $r=0,1,\ldots,n$, *suppose that the characteristic equation*

$$\phi(\mu) = a_0\mu^n+a_1\mu^{n-1}+\ldots+a_n=0$$

has l different roots $\mu_1, \mu_2, \ldots, \mu_l$ occurring $n_1, n_2, \ldots, n_l$ times respectively $(n_1+n_2+\ldots+n_l=n)$, and let n' be the largest of the numbers $n_1, n_2, \ldots n_l$.

If a function $\tau(x)$ exists such that, for sufficiently large values of x,

$$|\alpha_r(x)| \leqq \tau(x)/x^{2n'-2}, \qquad r=0, 1, \ldots, n,$$

then for the same values of x the equation $Y(x)=0$ has n linearly independent solutions $y_{i,k}(x)$ expressible in the form

$$y_{i,k}(x) = x^{k-1}\mu_i^x[1+\varepsilon_{i,k}(x), \qquad i=1, 2, \ldots, l; \ k=1, 2, \ldots, n_i,$$

where $\varepsilon_{i,k}(x)$ vanishes at infinity to at least as high an order as that of $T(x)$.

V. *The Non-homogeneous Equation.*

18. With the results of the previous section before us, it is a simple matter to determine the behavior of the complete solution of the non-homogeneous equation

$$Y(x) = X(x), \tag{41}$$

with proper restrictions on the right-hand member. This subject we shall discuss briefly in the present section.

---

[*] The function $\tau(x)$ is chosen in this way for simplicity. Cf. § 7.

Let the homogeneous equation $Y(x)=0$ corresponding to (41) satisfy the conditions of Theorem I, and suppose that $X(x)$ has the form

$$X(x)=\nu^x\,[b+\beta(x)]=\nu^x h(x),$$

where ν and b are constants different from 0 and $\beta(x)$ satisfies the conditions that are imposed by Theorem I upon the functions $\alpha_r(x)$.

We begin by transforming equation (41) into a homogeneous equation of order $n+1$ by the usual process. Equation (41) thus takes the form

$$\begin{aligned}&\mathbf{a}_0(x+1)h(x)y(x+n+1)+[\mathbf{a}_1(x+1)h(x)-\nu\,\mathbf{a}_0(x)h(x+1)]y(x+n)\\&\quad+[\mathbf{a}_2(x+1)h(x)-\nu\,\mathbf{a}_1(x)h(x+1)]y(x+n-1)+\ldots.\\&\quad+[\mathbf{a}_n(x+1)h(x)-\nu\mathbf{a}_{n-1}(x)h(x+1)]y(x+1)-\nu\mathbf{a}_n(x)h(x+1)y(x)=0.\end{aligned}\tag{42}$$

Thus if we write (42) in the form

$$\begin{aligned}&[a_0b+\gamma_0(x)]y(x+n+1)+[(a_1-a_0\nu)b+\gamma_1(x)]y(x+n)+\ldots.\\&\quad+[(a_n-a_{n-1})b+\gamma_n(x)]y(x+1)+[-a_n\nu\,b+\gamma_{n+1}(x)]y(x)=0,\end{aligned}\tag{43}$$

it appears that the functions $\gamma_0(x),\gamma_1(x),\ldots.,\gamma_{n+1}(x)$ satisfy all the conditions that have been imposed upon $\alpha_0(x)$, $\alpha_1(x),\ldots.,\alpha_n(x)$.

The characteristic equation

$$(\mu-\nu)\phi(\mu)=a_0\mu^{n+1}+(a_1-a_0\nu)\mu^n+\ldots.+(a_n-a_{n-1}\nu)\mu-a_n\nu=0$$

evidently has the roots $\mu_1,\ldots.,\mu_l,\nu$, where μ_r occurs n_r times, $r=1,\ldots.,l$.

19. Suppose first that

$$\nu\neq\mu_r,\qquad r=1,\ldots.,l.$$

Then by Theorem I, equation (43) has a particular solution $\bar{y}(x)$ expressible for large values of x in the form

$$\bar{y}(x)=c\,\nu^x[1+\varepsilon(x)],$$

where c is an arbitrary constant and $\bar{\varepsilon}(x)$ vanishes to at least as high an order as that of $T(x)$ when $x=\infty$. Further, since $\bar{y}(x)$ is evidently not a solution of the equation $Y(x)=0$, it must with proper choice of c satisfy equation (41). Substituting in (41), we find that $\bar{y}(x)$ will be a solution of that equation also, provided we take

$$c=\frac{b}{\phi(\nu)}.$$

20. Next, suppose ν is a root of the equation $\phi(\mu)=0$, say $\nu=\mu_h$. Two cases must be distinguished.

If $n_h<n'$, then equation (43) has by Theorem I a particular integral

$$\bar{y}(x)=c\,x^{n_h}\nu^x[1+\bar{\varepsilon}(x)].$$

Substituting in (41), we find that $\bar{y}(x)$ satisfies that equation provided

$$c=\frac{b}{\phi^{(n_h)}(\nu)}.$$

On the other hand, if $n_h = n'$, we must assume that for large x

$$|\gamma_r(x)| < \tau(x)/x^{2n'}, \qquad r=0, 1, \ldots, n+1.$$

In particular, this will happen if the functions $\alpha_0(x)$, $\alpha_1(x), \ldots, \alpha_n(x)$, $\beta(x)$ all satisfy this same condition. With this assumption the present case reduces to the one immediately preceding.

The results of this section, when combined with Theorem I, evidently give the behavior of the complete solution of (41).

PART II. LINEAR DIFFERENTIAL EQUATIONS.

VI. *The Fundamental Theorems.*

21. We shall make use of two general theorems on linear differential equations, which will now be stated.*

In the differential equation

$$y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots + a_n(x)\, y = 0, \tag{44}$$

suppose that the coefficients $a_1(x), \ldots, a_n(x)$, together with their first n derivatives, are continuous for all sufficiently large positive values of x. Let $z_1(x), \ldots, z_n(x)$ denote n auxiliary functions of x, which, together with their first n derivatives, are likewise continuous when x is large, and are such that the determinant

$$Q(x) = \begin{vmatrix} z_1(x) & z_1'(x) \ldots z_1^{(n-1)}(x) \\ z_2(x) & z_2'(x) \ldots z_2^{(n-1)}(x) \\ \ldots & \ldots\ldots\ldots\ldots \\ z_n(x) & z_n'(x) \ldots z_n^{(n-1)}(x) \end{vmatrix}$$

never vanishes. Let $A_r(x)$ be the minor of $Q(x)$ with respect to the element $z_r^{(n-1)}(x)$. Also put

$$Z_r(x) = z_r(x)\, a_n(x) - [z_r(x)\, a_{n-1}(x)]' + \ldots$$
$$+ (-1)^{n-1} [z_r(x)\, a_1(x)]^{(n-1)} + (-1)^n z_r^{(n)}(x),$$

and denote by $q(x, t)$ the determinant formed from $Q(x)$ by replacing the elements $z_r^{(n-1)}(x)$ by $Z_r(t)$, $r=1, \ldots, n$.

Place

$$K(x,t) = (-1)^{n-1} q(x,t)/Q(x),$$
$$f_r(x) = u_{r,0}(x) = v_{r,0}(x) = (-1)^{n-1} C_r A_r(x)/Q(x), \qquad r=1, \ldots, n,$$

where C_r is an arbitrary constant,

$$u_{r,m}(x) = \int_x^\infty K(x,t)\, u_{r,m-1}(t)\, dt, \qquad m=1, 2, \ldots,$$
$$v_{r,m}(x) = \int_{x_0}^x K(x,t)\, v_{r,m-1}(t)\, dt, \qquad m=1, 2, \ldots.$$

---

* These theorems, which differ only slightly from those of Dini (*loc. cit.*), have been stated and proved in a previous paper by the present writer (AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVI (1914), pp. 151–166. They are restated here for convenience of reference.

Then we have

THEOREM C. *If for all values of* $x \geqq x_0$

(a) *the series*

$$y_i(x) = \sum_{m=0}^{\infty} u_{i,m}(x) \tag{45}$$

converges;

(b) *the series for* $y_i(t)$, *when multiplied by* $K(x,t)$, *may be integrated term by term with respect to t from x to* ∞ *;*

(c) *each of the integrals*

$$\int_x^\infty y_i(t)\, Z_r(t)\, dt, \qquad r=1, \ldots., n,$$

has a meaning,

then for such values of x the function $y_i(x)$ *is an integral of* (44).

THEOREM D. *If for all values of* $x \geqq x_0$

(a) *the series*

$$y_i(x) = \sum_{m=0}^{\infty} v_{i,m}(x)$$

converges;

(*b*) *the series for* $y_i(t)$, *when multiplied by* $K(x,t)$, *may be integrated term by term with respect to t from* x_0 *to x,*

then for such values of x the function $y_i(x)$ *is an integral of* (44).

VII. *The Homogeneous Equation: Multiple Roots.*

22. Since the case of distinct roots is included in that of multiple roots, we pass at once to the general case.

Consider the differential equation

$$Y(x) = y^{(n)} + [a_1 + \alpha_1(x)]\, y^{(n-1)} + \ldots. + [a_n + \alpha_n(x)]\, y = 0, \tag{1}$$

where the functions $\alpha_1(x), \ldots., \alpha_n(x)$, together with their first n derivatives, are continuous for large values of x.

Let $\tau(x)$ denote one of the set of functions

$$\frac{1}{x^\nu}, \ \frac{1}{x\,(\log x)^\nu}, \ \frac{1}{x \log x\,(\log\log x)^\nu}, \ \ldots., \qquad \nu > 1,$$

and set

$$T(x) = \int_x^\infty \tau(t)\, dt.$$

Suppose that the characteristic equation

$$\phi(\mu) = \mu^n + a_1 \mu^{n-1} + \ldots. + a_n = 0 \tag{46}$$

corresponding to (1) has l distinct roots $\mu_1, \mu_2, \ldots., \mu_l$ occurring $n_1, \ldots., n_l$ times respectively $(n_1 + \ldots. + n_l = n)$. Assume further that, for large values of x, a function $\tau(x)$ can be chosen such that

$$|\alpha_r^{(s)}(x)| \leqq \tau(x)/x^{2n'-2}, \qquad r=1,\ldots,n;\ s=0,1,\ldots,2n-1,$$

where n' is the largest of the numbers $n_1,\ldots,n_l$.

Choose n auxiliary functions of x of the form*

$$z_{r,q}(x)=x^{n_r-q}e^{-\mu_r x}, \qquad r=1,\ldots,l;\ q=1,\ldots,n_r.$$

Placing

$$\delta=\prod_{r,s} 0!\,1!\ldots(n_r-1)!\,(\mu_r+\mu_s)^{n_r n_s}, \qquad r=1,\ldots,l;\ r>s,$$

we find

$$Q(x)=\delta e^{-\mu x},$$

where

$$\mu=n_1\mu_1+n_2\mu_2+\ldots+n_l\mu_l,$$

so that the hypothesis $Q(x)\neq 0$ is satisfied. Also

$$Z_{r,q}(x)=x^{n_r-q}e^{-\mu_r x}\rho_{r,q}(x),$$

where a constant M_1 can be found such that, for sufficiently large x,

$$|\rho_{r,q}(x)| \leqq M_1\tau(x)/x^{2n'-2}.$$

Further, with proper choice of $C_{r,q}$, we have

$$f_{r,q}(x)=x^{q-1}e^{\mu_r x};$$

also

$$K(x,t)=\sum_{r=1}^{l}\sum_{q=1}^{n_r} x^{q-1}e^{\mu_r(x-t)}t^{n_r-q}\sigma_{r,q}(t),$$

where $\sigma_{r,q}(t)$ is a constant multiple of $\rho_{r,q}(t)$.

Let the roots $\mu_1,\ldots,\mu_l$ of (46) be so arranged that

$$R[\mu_1]\leqq R[\mu_2]\leqq\ldots\leqq R[\mu_l],$$

where $R[\mu]$ means the real part of μ, and suppose in the first place that these real parts are all distinct.

23. Let us form as trial solutions of (1) a group of functions $y_{1,k}(x)$, $k=1,\ldots,n_1$, as given by (45).

We proceed to prove by mathematical induction that

$$|u_{1,k,m}(x)| \leqq [M_1 n T(x)]^m\,|f_{1,k}(x)|, \qquad k=1,\ldots,n_1. \tag{47}$$

This relation is true for $u_{1,k,0}(x)$ by definition. Assume it to hold for $u_{1,k,m-1}(x)$. Then

$$\begin{aligned}|u_{1,k,m}(x)| &\leqq \int_x^\infty |K(x,t)\,u_{1,k,m-1}(t)|\,dt\\ &\leqq (M_1 n)^{m-1}\int_x^\infty [T(t)]^{m-1}\,|K(x,t)\,f_{1,k}(t)|\,dt.\end{aligned} \tag{48}$$

---

* Cf. foot-note, p. 67.

But

$$
\begin{aligned}
|K(x,t)f_{1,k}(t)| &\leqq |f_{1,k}(x)| \sum_{r=1}^{l} \sum_{q=1}^{n_r} x^{q-k} |e^{(\mu_r-\mu_1)(x-t)}| t^{n_r-q+k-1} |\sigma_{r,q}(t)| \\
&\leqq |f_{1,k}(x)| M_1 \tau(t) \sum_{r=1}^{l} \sum_{q=1}^{n_r} |e^{(\mu_r-\mu_1)(x-t)}| x^{q-k} t^{n_r-q+k-2n'+1} \\
&\leqq |f_{1,k}(x)| M_1 \tau(t) \sum_{r=1}^{l} \sum_{q=1}^{n_r} |e^{(\mu_r-\mu_1)(x-t)}| x^{1-k} \left(\frac{x}{t}\right)^{q-1} t^{n_r+k-2n'} \\
&\leqq |f_{1,k}(x)| M_1 n \tau(t),
\end{aligned}
\tag{49}
$$

since

$$R[\mu_r-\mu_1] \geqq 0, \quad x \leqq t, \quad k \geqq 1, \quad q \geqq 1, \quad 2n' \geqq n_r+k.$$

Substituting (49) in (48), we find

$$|u_{1,k,m}(x)| \leqq (M_1 n)^m |f_{1,k}(x)| \int_x^\infty [T(t)]^{m-1} \tau(t)\, dt. \tag{50}$$

Now the function $T(x)$ decreases monotonically as x increases, so that in the right member of (50) we may replace $T(t)$ by its greatest value $T(x)$. Whence

$$|u_{1,k,m}(x)| \leqq (M_1 n)^m [T(x)]^{m-1} |f_{1,k}(x)| \int_x^\infty \tau(t)\, dt,$$

and formula (47) is established.

Let x now be taken so large that

$$M_1 n T(x) \leqq \theta, \qquad 0<\theta<1, \tag{51}$$

where θ is a constant independent of x. Then by (47) we have

$$\sum_{m=0}^{\infty} |u_{1,k,m}(x)| \leqq \frac{|f_{1,k}(x)|}{1-M_1 n T(x)} \leqq \frac{|f_{1,k}(x)|}{1-\theta}. \tag{52}$$

Thus condition (a) of Theorem C is satisfied.

To show that (b) is satisfied we make use of the following theorem:*

"If $\Sigma f_m(x)$ converges uniformly in any fixed interval $a \leqq x \leqq b$, where b is arbitrary, and if $\phi(x)$ is continuous for all finite values of x, then

$$\int_a^\infty \phi(x) \Sigma f_m(x)\, dx = \Sigma \int_a^\infty \phi(x) f_m(x)\, dx,$$

provided that the integral $\int_a^\infty |\phi(x)| \Sigma |f_m(x)|\, dx$ is convergent."

Let us take $a=x$, $x=t$, $\phi(t)=K(x,t)$, $f_m(t)=u_{1,k,m}(t)$. The uniform convergence of the series $\sum_{m=0}^{\infty} u_{1,k,m}(t)$ in any interval $x \leqq t \leqq b$ follows at once from (47) and (51), by Weierstrass's test. The continuity of the functions $\sigma_{r,q}(t)$, and hence that of $K(x,t)$, follows from our assumptions regarding the continuity of the coefficients $\alpha_1(x), \ldots, \alpha_n(x)$ and their derivatives. And the convergence of the integral $\int_x^\infty |K(x,t)| \sum_{m=0}^{\infty} |u_{1,k,m}(t)|\, dt$ is a consequence of (52) and (49).

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* Bromwich, "Infinite Series" (1908), p. 453.

Finally, we note that

$$\begin{aligned}|y_{1,k}(x)\,Z_{r,q}(x)| &\leqq |Z_{r,q}(x)|\,.\sum_{m=0}^{\infty}|u_{1,k,m}(x)|\\ &\leqq \frac{1}{1-\theta}M_1\,x^{k+1+n_r-q-2n'}\,|e^{(\mu_1-\mu_r)x}|\,\tau(x)\\ &\leqq \frac{M_1\tau(x)}{1-\theta}.\end{aligned}$$

Thus (c) holds, and the conditions of Theorem C are satisfied by the functions $y_{1,k}(x)$, $k=1,\ldots.,n_1$.

We have now proved that these functions are integrals of (1) for values of x so large that (51) holds. Further, by (47), we have

$$\begin{aligned}|y_{1,k}(x)-f_{1,k}(x)| &\leqq \sum_{m=1}^{\infty}|u_{1,k,m}(x)|\\ &\leqq \frac{M_1 n\,T(x)\,|f_{1,k}(x)|}{1-\theta}.\end{aligned}$$

Thus we may write

$$y_{1,k}(x)=x^{k-1}e^{\mu_1 x}[1+\varepsilon_{1,k}(x)],\qquad k=1,\ldots.,n_1, \tag{53}$$

where $\varepsilon_{1,k}(x)$ vanishes with increasing x to at least as high an order as that of $T(x)$.

24.* We shall prove next that

$$\lim_{x=\infty} y_{1,k}^{(s)}(x)/y_{1,k}(x)=\mu_1^s,\qquad s=1,\ldots.,n;$$

or in other words, that we may write

$$y_{1,k}^{(s)}(x)=x^{k-1}e^{\mu_1 x}[\mu_1^s+\zeta_{1,k,s}(x)], \tag{54}$$

where

$$\lim_{x=\infty}\zeta_{1,k,s}(x)=0.$$

To do this, let

$$K_g(x,t)=\frac{\partial^g}{\partial x^g}K(x,t),$$

$$K_g^{(h)}(x,x)=\frac{d^h}{dx^h}\{[K_g(x,t)]_{t=x}\}.$$

It can be shown by elementary calculus, as a result of our hypotheses concerning $\alpha_1(x),\ldots.,\alpha_n(x)$ and their derivatives, that, if $h\leqq n-1$,

$$|K_g^{(h)}(x,x)|\leqq M_2\tau(x)/x^{n'-1}, \tag{55}$$

where M_2 is a sufficiently large constant.

We note that, as a result of the conditions of Theorem C, the function $y_{1,k}(x)$ satisfies the singular Volterra equation

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* The present section, being relatively unimportant, will be condensed as much as possible. The missing details are readily supplied.

$$y_{1,k}(x) = f_{1,k}(x) + \int_x^\infty K(x,t)\, y_{1,k}(t)\, dt. \tag{56}$$

Differentiating (56)* $s-1$ times in succession ($s \leqq n-1$), we find

$$y_{1,k}^{(s-1)}(x) = f_{1,k}^{(s-1)}(x) + S_1 y_{1,k}^{(s-2)}(x) + S_2 y_{1,k}^{(s-3)}(x) + \ldots\ldots + S_{s-1} y_{1,k}(x) \\ + \int_x^\infty K_{s-1}(x,t) y_{1,k}(t)\, dt, \tag{57}$$

where S_1, S_2, ..., S_{s-1} are linear and homogeneous in the derivatives $K_g^{(h)}(x, x)$, so that by virtue of (55) constants M_3 and M_4 can be found such that

$$|S_i(x)| \leqq M_3 \tau(x)/x^{n'-1}, \qquad i=1, \ldots., s-1, \tag{58}$$

$$|S_i'(x)| \leqq M_4 \tau(x)/x^{n'-1}, \qquad i=1, \ldots., s-1. \tag{59}$$

We are now able to prove (54) by induction. The formula is known to hold for $y_{1,k}(x)$. Let it be supposed true for

$$y_{1,k}'(x),\ y_{1,k}''(x),\ \ldots.,\ y_{1,k}^{(s-1)}(x), \qquad s \leqq n-1.$$

Differentiating (57), we find

$$y_{1,k}^{(s)}(x) = f_{1,k}^{(s)}(x) + S_1 y_{1,k}^{(s-1)}(x) + (S_1' + S_2) y_{1,k}^{(s-2)}(x) + \ldots\ldots + (S_{s-2}' + S_{s-1})\, y_{1,k}'(x) \\ + S_{s-1}' y_{1,k}(x) - K_{s-1}(x,x)\, y_{1,k}(x) + \int_x^\infty K_s(x,t)\, y_{1,k}(t)\, dt. \tag{60}$$

Denoting by $x^{k-1}\,|e^{\mu_1 x}|\, W(x)$ the sum of the absolute values of all the terms in the right member of (60) except the first and last, we have by virtue of (55), (58) and (59)

$$\lim_{x=\infty} W(x) = 0.$$

Evidently

$$\lim_{x=\infty} x^{1-k}\, e^{-\mu_1 x} f_{1,k}^{(s)}(x) = \mu_1^s.$$

As regards the last term of (60), we find by inspecting the form of $K_s(x,t)$ that

$$\left|\int_x^\infty K_s(x,t)\, y_{1,k}(t)\, dt\right| \leqq \int_x^\infty |K_s(x,t)\, y_{1,k}(t)|\, dt \\ \leqq M_5 \int_x^\infty |K(x,t) f_{1,k}(t)|\, dt,$$

where M_5 is a certain constant. But by (49) this becomes

$$\left|\int_x^\infty K_s(x,t)\, y_{1,k}(t)\, dt\right| \leqq M_1 M_5 n\, x^{k-1}\, |e^{\mu_1 x}|\, T(x).$$

We have thus proved that

$$\lim_{x=\infty} x^{1-k}\, e^{-\mu_1 x} y_{1,k}^{(s)}(x) = \mu_1^s, \qquad s=1, \ldots., n-1.$$

For the case $s=n$, we have directly from (1) that

$$\lim_{x=\infty} x^{1-k}\, e^{-\mu_1 x} y_{1,k}^{(n)}(x) = -a_1 \mu_1^{n-1} - a_2 \mu_2^{n-2} - \ldots. - a_n = \mu_1^n.$$

Thus (54) is established.

25. If we try to apply the process of § 23 to find a group of integrals $y_{2,k}$ of (1) corresponding to the root μ_2, we find that some of the improper integrals occurring diverge because of the fact that $R[\mu_2 - \mu_1] > 0$. We shall therefore employ Theorem D, first making a change in our system of auxiliary functions.

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* Cf. Goursat-Hedrick, "Mathematical Analysis," p. 370.

The differential equation

$$Z(x)=[a_n+\alpha_n(x)]\,z(x)-\frac{d}{dx}\{[a_{n-1}+\alpha_{n-1}(x)]\,z(x)\}+\dots$$
$$+(-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\{[a_1+\alpha_1(x)]\,z(x)\}+(-1)^n\frac{d^n}{dx^n}z(x)=0$$

evidently satisfies all the conditions that have been imposed upon equation (1). It therefore has a group of solutions, given by (53), corresponding to that root of its characteristic equation having the least real part. The roots are evidently those of (46) with signs changed, so that the solutions in question may be put in the form

$$z_{l,q}(x)=x^{n_l-q}\,e^{-\mu_l x}\,[1+\eta_{l,q}(x)],\qquad q=1,\dots,n_l. \tag{61}$$

Upon introducing the functions $z_{l,q}(x)$ as given by (61) in our auxiliary system in place of $z_{l,q}(x)$ as originally chosen, we find that, in $K(x,t)$, r takes only the values $1,\dots,l-1$. We are now able to show by Theorem D that (1) is satisfied by each of the functions

$$y_{l,k}(x)=\sum_{m=0}^{\infty}v_{l,k,m}(x),\qquad k=1,\dots,n_l,$$

where, in $v_{l,k,m}(x)$, the functions $f_{l,k}(x)$ and $K(x,t)$ must of course be formed in terms of our revised auxiliary system.

First, we have

$$f_{r,q}(x)=x^{q-1}\,e^{\mu_r x}\,\lambda_{r,q}(x),$$

where it follows from (54) that $\lambda_{r,q}(x)$ remains less than some fixed constant for large values of x. Hence,

$$|K(x,t)\,f_{l,k}(t)|\leqq\sum_{r=1}^{l-1}\sum_{q=1}^{n_r}x^{q-1}\,|e^{\mu_r(x-t)+\mu_l t}|\,t^{n_r-q+k-1}|\nu_{r,q}(t)|,$$

where a constant M_6 can be found such that

$$|\nu_{r,q}(x)|\leqq M_6\tau(x)/x^{2n'-2}.$$

Thus

$$|K(x,t)\,f_{l,k}(t)|\leqq M_6\tau(t)\,|f_{l,q}(x)|\sum_{r=1}^{l-1}\sum_{q=1}^{n_r}|e^{(\mu_r-\mu_l)(x-t)}|\,x^{q-k}\,t^{n_r-q+k-2n'+1}.$$

We shall now prove by induction that

$$|v_{l,k,m}(x)|\leqq[M_6(n-1)x\tau(x)]^m|f_{l,k}(x)|.$$

This is evidently true of $v_{l,k,0}(x)$. Suppose it to hold for $v_{l,k,m-1}(x)$. Then

$$\begin{aligned}|v_{l,k,m}(x)|&\leqq\int_{x_0}^{x}|K(x,t)\,v_{l,k,m-1}(t)|\,dt\\ &\leqq[M_6(n-1)]^{m-1}\int_{x_0}^{x}[t\tau(t)]^{m-1}|K(x,t)\,f_{l,k}(t)|\,dt\\ &\leqq M_6^m(n-1)^{m-1}|f_{l,k}(x)|\int_{x_0}^{x}[t\tau(t)]^{m-1}\tau(t)\sum_{r=1}^{l}\sum_{q=1}^{n_r}|e^{(\mu_l-\mu_r)(t-x)}|\,x^{q-k}\,t^{n_r-q+k-2n'+1}\,dt\\ &\leqq M_6^m(n-1)^{m-1}|f_{l,k}(x)|\int_{x_0}^{x}[t\tau(t)]^{m-1}\tau(t)\sum_{r=1}^{l-1}\sum_{q=1}^{n_r}|e^{(\mu_l-\mu_r)(t-x)}|\,x^{q-k}\,t^{k-q}\,dt,\end{aligned}$$

since $2n' \geqq n_r + 1$. Now it results from the form of $\tau(x)$ that each of the functions $e^{R[\mu_l - \mu_r]t}\, t^{m-1-q+k}\, [\tau(t)]^m$, $r=1, \ldots, l-1$, increases monotonically with t. We may therefore replace each function by its maximum value, *i. e.*, its value when $t=x$, in the right member of the inequality last written. Thus

$$|v_{l,k,m}(x)| \leqq [M_6(n-1)\,\tau(x)]^m x^{m-1}\, |f_{l,k}(x)| \int_{x_0}^{x} dx$$
$$< [M_6(n-1)\, x\tau(x)]^m\, |f_{l,k}(x)|,$$

which is the required result.

It can now be shown without difficulty that for sufficiently large values of x the functions $y_{l,k}(x)$ satisfy the conditions of Theorem D, and therefore satisfy (1); also that we may write

$$y_{l,k}(x) = x^{k-1} e^{\mu_l x} [1 + \varepsilon_{l,k}(x)], \qquad k=1, \ldots, n_l.$$

The result for $y_{l,k}^{(s)}(x)$ analogous to (54) is also readily established.

26. The remainder of the process is precisely like that used at the corresponding stage of the work on difference equations (§ 11). We find ultimately n linearly independent integrals of (1), of the form indicated in the theorem below.

The modification required in case the real parts of the roots $\mu_1, \mu_2, \ldots, \mu_l$ are not all distinct is also obvious. In particular the case treated by Dini, in which the roots all have the same real part, yields to Theorem A at once.

27. We summarize in

THEOREM II. *Let $\tau(x)$ be one of the system of functions*[*]

$$\frac{1}{x^\nu}, \quad \frac{1}{x(\log x)^\nu}, \quad \frac{1}{x \log x (\log\log x)^\nu}, \quad \ldots, \qquad \nu > 1,$$

and put

$$T(x) = \int_x^\infty \tau(t)\, dt.$$

In the differential equation

$$Y(x) = y^{(n)} + [a_1 + \alpha_1(x)]\, y^{(n-1)} + \ldots + [a_n + \alpha_n(x)]\, y = 0,$$

suppose that, when x is large, the functions $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)$ are continuous and possess $2n-1$ continuous derivatives. Suppose also that the characteristic equation

$$\phi(\mu) = \mu^n + a_1 \mu^{n-1} + \ldots + a_n = 0$$

has l different roots $\mu_1, \mu_2, \ldots, \mu_l$, occurring $n_1, n_2, \ldots, n_l$ times respectively $(n_1 + n_2 + \ldots + n_l = n)$, and let n' be the largest of the numbers $n_1, n_2, \ldots, n_l$.

If a function $\tau(x)$ exists such that for sufficiently large values of x

$$|\alpha_r^{(s)}(x)| \leqq \tau(x)/x^{2n'-2}, \qquad r=0, 1, \ldots, n;\ s=0, 1, \ldots, 2n-1,$$

---

[*] The function $\tau(x)$ is chosen in this way merely for simplicity. Cf. § 7.

then for the same values of x *the equation* $Y(x)=0$ *has* n *linearly independent solutions* $y_{i,k}(x)$ *expressible in the form*

$$y_{i,k}(x)=x^{k-1}e^{\mu_i x}[1+\varepsilon_{i,k}(x)], \qquad i=1,2,\ldots,l;\ k=1,2,\ldots,n_i,$$

where $\varepsilon_{i,k}(x)$ *vanishes at infinity to at least as high an order as that of* $T(x)$, *while further*

$$y_{i,k}^{(s)}(x)=x^{k-1}e^{\mu_i x}[\mu_i^s+\zeta_{i,k,s}(x)], \qquad s=1,2,\ldots,n,$$

where

$$\lim_{x=\infty}\zeta_{i,k,s}(x)=0.$$

VIII. *The Non-homogeneous Equation.*

28. In the equation

$$Y(x)=X(x), \tag{62}$$

let the right-hand member have the form

$$X(x)=e^{\nu x}[b+\beta(x)],$$

where ν and b are constants and $b\neq 0$. Let the functions $\alpha_1(x)$, $\alpha_2(x)$, ..., $\alpha_n(x)$, $\beta(x)$ be continuous and possess $2n+1$ continuous derivatives, and suppose a function $\tau(x)$ exists such that

$$|\alpha_r^{(s)}(x)|\leqq\tau(x)/x^{2n'-2}, \qquad r=1,\ldots,n;\ s=0,1,\ldots,2n+1,$$
$$|\beta^{(s)}(x)|\leqq\tau(x)/x^{2n'-2}, \qquad s=0,1,\ldots,2n+1.$$

The work of determining by Theorem II the behavior of a particular solution of (62) is so similar to that of §§ 18–20 that we give only the results.

If $\nu\neq\mu_r$, $r=1,\ldots,l$, equation (62) has a particular integral of the form

$$\bar{y}=\frac{b}{\phi(\nu)}e^{\nu x}[1+\bar{\varepsilon}(x)],$$

where $\bar{\varepsilon}(x)$ vanishes at infinity to as high an order as that of $T(x)$.

If $\nu=\mu_h$, where $n_h<n'$, equation (62) has a particular integral

$$\bar{y}=\frac{b}{\phi^{(n_h)}(\nu)}x^{n_h}e^{\nu x}[1+\bar{\varepsilon}(x)]. \tag{63}$$

If $\nu=\mu_h$, where $n_h=n'$, equation (62) has still an integral of the form (63), provided we assume that

$$|\alpha_r^{(s)}(x)|\leqq\tau(x)/x^{2n'}, \qquad r=1,\ldots,n;\ s=0,1,\ldots,2n+1,$$
$$|\beta^{(s)}(x)|\leqq\tau(x)/x^{2n'}, \qquad s=0,1,\ldots,2n+1.$$

The results of this section, together with Theorem II, determine the behavior of the complete integral of (62).

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